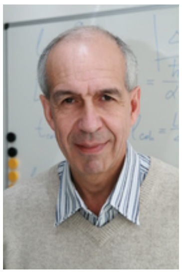
Quantum mechanics aspects and subtleties of neutrino oscillations

E.K. Akhmedov
Max-Planck-Institut für Kernphysik, Saupfercheckweg 1,
69117 Heidelberg, Germany



Neutrino oscillations appear to be a simple quantum mechanical phenomenon. However, a closer look at them reveals a number of subtle points and apparent paradoxes. Some of the basic issues of the theory of neutrino oscillations are still being debated. I discuss, from a historical perspective, how the role of quantum mechanical aspects of neutrino oscillations was realized in the course of the development of the theory of this phenomenon.

1 Neutrino Oscillations and Quantum Mechanics

I will discuss how the various quantum mechanical (QM) aspects of neutrino oscillations were realized in the course of the development of the theory of this phenomenon. I will also use this opportunity to revisit some subtle issues of this theory and discuss their resolution.

Neutrino oscillations is a periodic change of neutrino flavour. Particles usually change their identity in collisions with other particles or when they decay, so it may look strange that neutrinos can change their flavour without any external influence. However, the phenomenon of oscillations is actually well known in quantum mechanics. A textbook example is a 2-level quantum system. If one produces the system in one of its stationary states, $|\Psi_1\rangle$ or $|\Psi_2\rangle$, its time evolution is very simple: $|\Psi(t)\rangle = e^{-iE_1t}|\Psi_1\rangle$ or $|\Psi(t)\rangle = e^{-iE_2t}|\Psi_2\rangle$, respectively. The probability for the system to remain in such a state does not change with time. However, if the system is prepared in a state that is a linear superposition of its stationary states,

$$|\Psi(0)\rangle = a|\Psi_1\rangle + b|\Psi_2\rangle\,, \qquad (|a|^2 + |b|^2 = 1)\,, \tag{1}$$

its time evolution is more complex:

$$|\Psi(t)\rangle = ae^{-i\,E_1\,t}\,|\Psi_1\rangle + b\,e^{-i\,E_2\,t}\,|\Psi_2\rangle\,. \tag{2}$$

The probability that the system will be found in its initial state $|\Psi(0)\rangle$ at time t is

$$P_{\rm surv} = |\langle\Psi(0)|\Psi(t)\rangle|^2 = \big||a|^2\,e^{-i\,E_1\,t} + |b|^2\,e^{-i\,E_2\,t}\big|^2 \tag{3}$$

$$= 1 - 4|a|^2|b|^2\sin^2[(E_2 - E_1)\,t/2]\,. \tag{4}$$

It oscillates with time with the frequency $(E_2 - E_1)/2$ and the amplitude that takes its maximum value when $|a| = |b|$ ("maximum mixing") and vanishes when either a or b is zero (no mixing). This analogy gives a rather accurate description of the physical essence of neutrino oscillations, as in the charged-current processes neutrinos are produced (and detected) as flavour eigenstates, which are non-trivial linear superpositions of the eigenstates of free propagation (mass eigenstates). The above description in terms of evolution in time of a superposition of stationary states was actually used in most of the early papers on neutrino oscillations [1,2,3,4,5]. It, however, leaves out the question of how neutrino flavour changes with distance traveled by the neutrinos (see the discussion of the "time to space conversion" procedure below, though).

1.1 Tricky Issues

Alhough neutrino oscillations appear to be a simple QM phenomenon, a closer look at them reveals a number of subtle points and apparent paradoxes. A number of fundamental issues of the theory of neutrino oscillations have been actively debated ever since the idea of the oscillations was put forward by Pontecorvo [6,7a]. These include

- Do neutrino mass eigenstates composing a given flavour eigenstate have same energy or same momentum?
- Can one use plane waves or stationary states for describing neutrino oscillations?
- Do the oscillations contradict energy-momentum conservation?
- Under what conditions can the oscillations be observed?
- Is wave packet approach necessary for describing neutrino oscillations?
- When are the oscillations described by a universal (i.e. production– and detection–independent) probability?
- Is the standard oscillation formula correct? What is its domain of applicability?
- How to get correctly normalized oscillation probabilities?
- Are the oscillation probabilities Lorentz invariant?
- Why do we say that charged leptons are produced as mass eigenstates and neutrinos as flavour states and not the other way around?
- Do neutrinos produced in $\pi \to l\nu_l$ decays oscillate when the charged lepton is not detected?
- Do charged leptons oscillate?

I will now briefly discuss how quantum mechanics allows us to answer these questions.

2 Master Formula for the Probability of Neutrino Oscillations in Vacuum

In the standard approach to neutrino oscillations the state vector describing a flavour eigenstate neutrino ν_α $(\alpha = e, \mu, \tau, ...)$ is considered to be a linear superposition of the state vectors of the mass eigenstate neutrinos ν_i $(i = 1, 2, 3, ...)$:

$$|\nu_\alpha\rangle \;=\; \sum_i U^*_{\alpha i}\,|\nu_i\rangle\,, \tag{5}$$

[a] The papers debating the basics of the neutrino oscillation theory are too plentiful to be cited here. An incomplete (but representative) list of references can be found on slide 5 of the presentation slides of this talk at http://neutrinohistory2018.in2p3.fr/programme.html.

where U is the leptonic mixing matrix. From this expression one can derive the master formula for the probability of $\nu_\alpha \to \nu_\beta$ oscillations in vacuum:

$$P_{\alpha\beta}(L) = \left| \sum_i U_{\beta i}\, e^{-i\frac{\Delta m^2_{ik}}{2p}L}\, U^*_{\alpha i} \right|^2 . \tag{6}$$

How is it usually obtained?

2.1 Simplified Derivations: Same Energy and Same Momentum Approaches

Derivations of the oscillation probability which can be found in many texts typically proceed as follows. The state vector describing a flavour eigenstate neutrino ν_α produced at time $t = 0$ and coordinate $\vec{x} = 0$ is taken to be given by Eq. (5). Assuming that neutrinos are described by plane waves, the evolved neutrino state after time t at the position $\vec{x}$ is then

$$|\nu(t, \vec{x})\rangle \;=\; \sum_i U^*_{\alpha i}\, e^{-ip_i x} |\nu_i^{\rm mass}\rangle\,. \tag{7}$$

That is, each mass eigenstate ν_i picks up the phase factor $e^{-i\phi_i}$, where

$$\phi_i \;\equiv\; p_i\, x \;=\; E_i t - \vec{p}_i\, \vec{x}\,. \tag{8}$$

Projecting the evolved neutrino state on the flavour eigenstate ν_β and taking the squared modulus yields the oscillation probability

$$P(\nu_\alpha \to \nu_\beta;\, t, \vec{x}) \;=\; |\langle \nu_\beta | \nu(t, \vec{x})\rangle|^2\,. \tag{9}$$

To evaluate it, one needs to calculate the phase differences between different mass eigenstates (the oscillation phases) $\Delta\phi_{ik}$:

$$\Delta\phi \;=\; \Delta E \cdot t - \Delta\vec{p} \cdot \vec{x}\,. \tag{10}$$

Here the subscripts ik are omitted from $\Delta\phi$, ΔE and $\Delta\vec{p}$ in order to simplify the notation. Clearly, different neutrino mass eigenstates composing a given flavour state cannot simultaneously have the same energy and the same momentum, as otherwise they would have had the same mass. Therefore in many studies two simplified approaches were adopted:

(a) Same momentum approach. Assume that all the mass eigenstates composing the produced neutrino flavour state have the same momentum, i.e. $\Delta\vec{p} = 0$. Then Eq. (10) gives $\Delta\phi = \Delta E \cdot t$, and the oscillation probability (9) depends only on the evolution time t. Since for ultra-relativistic neutrinos $E_i = \sqrt{\vec{p}^{\,2} + m_i^2} \simeq p + \frac{m_i^2}{2p}$, for the oscillation phase one finds

$$\Delta\phi = \Delta E \cdot t \simeq \frac{\Delta m^2}{2p} t\,. \tag{11}$$

Experimentally, the distance between the neutrino source and detector $L = |\vec{x}|$ rather than time of flight t is normally known. It is then usually argued that, as neutrinos propagate with nearly the speed of light,

$$L \simeq t\,, \tag{12}$$

and so one can replace $t \to L$ in Eq. (11) (in the literature on neutrino oscillations this procedure is sometimes called "time to space conversion"). With this replacement Eq. (11) yields the usual oscillation phase, and using it in Eq. (9) leads to the standard oscillation probability (6).

(b) Same energy approach. Assume now that all the mass eigenstates composing the produced neutrino flavour state have the same energy, i.e. $\Delta E = 0$. Eq. (10) then gives $\Delta\phi = -\Delta\vec{p}{\cdot}\vec{x}$. Assuming $\vec{x}||\vec{p}$ (which is well justified when the distance between the neutrino source and detector is large compared to their transverse sizes) one finds that the oscillation probability (9) depends

only on the distance L. For ultra-relativistic neutrinos one has $p_i = \sqrt{E^2 - m_i^2} \simeq E - \frac{m_i^2}{2E}$, and the oscillation phase is

$$\Delta\phi = -\Delta p \cdot L \simeq \frac{\Delta m^2}{2E} L \,. \tag{13}$$

This is the standard expression for the oscillation phase, which depends on the distance L traveled by neutrinos; unlike in the "same momentum" approach discussed above, it was not necessary to invoke the "time to space conversion" procedure to arrive at it. The resulting oscillation probability is again that of Eq. (6).

The above two approaches are very simple and transparent, and allow one to quickly get the desired result. The trouble with them is that they are both wrong.

The point is that there is no reason whatsoever to expect the neutrino mass eigenstates composing a flavour state to have either the same energy or the same momentum. These assumptions actually contradict energy-momentum conservation. This was first demonstrated by R. Winter [8], who considered neutrino emission in orbital electron capture by nuclei – a process with 2-body final state and simple kinematics. Another process with 2-body final state – charged pion decay – was discussed in this context by Giunti and Kim [9]. Let us follow their argument.

For a $\pi \to \mu\nu$ decay at rest, 4-momentum conservation gives for the energy and momentum of the produced neutrino mass eigenstate ν_i with mass m_i

$$E_i^2 \;=\; \frac{m_\pi^2}{4}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 + \frac{m_i^2}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) + \frac{m_i^4}{4m_\pi^2}\,, \tag{14}$$

$$p_i^2 \;=\; \frac{m_\pi^2}{4}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 - \frac{m_i^2}{2}\left(1 + \frac{m_\mu^2}{m_\pi^2}\right) + \frac{m_i^4}{4m_\pi^2}\,. \tag{15}$$

Neglecting terms of order m_i^4, one finds

$$E_i \;\simeq\; E + \xi\frac{m_i^2}{2E}\,, \qquad p_i \;\simeq\; E - (1-\xi)\frac{m_i^2}{2p}\,, \tag{16}$$

where

$$E \simeq p \equiv \frac{m_\pi}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \simeq 30 \text{ MeV}\,, \qquad \xi \;\equiv\; \frac{1}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \;\approx\; 0.2. \tag{17}$$

As can be seen from Eq. (16), same energy and same momentum assumptions correspond to $\xi = 0$ and $\xi = 1$, respectively; in reality, however, ξ is neither 0 nor 1 but about 0.2. Moreover, if we considered the $\pi \to e\nu$ decay rather than the $\pi \to \mu\nu$ one, we would have to replace m_μ by m_e in Eqs. (14)-(17); for the parameter ξ this would give $\xi \approx 0.5$, which is just in the middle between the values corresponding to same energy and same momentum assumptions.

Note that the results presented here will require modifications once the intrinsic QM uncertainties of the energies and momenta of all the particles participating in neutrino production are taken into account. However, the main conclusion remains unchanged: same energy and same momentum assumptions contradict kinematics and are in general incorrect.

2.2 Same E and Same p Approaches: More Problems

The internal inconsistencies of the "same energy" and "same momentum" approaches to neutrino oscillations can actually be seen even without invoking energy-momentum conservation. In the same momentum approach is is assumed that neutrinos have well-defined momentum, i.e. they are described by plane waves. However, the probability to find a particle described by a plane wave has no coordinate dependence, i.e. it is the same at any point in space. This means that propagation of neutrinos in space cannot be accounted for in this case. For neutrino oscillation experiments it is crucial that neutrinos are produced and detected in distinct regions of space, the distance between which is the experimental baseline L. However, in the plane-wave approach one cannot even define the neutrino production and detection regions.

Moreover, the oscillation phase in this case depends only on time (see Eq. (11)). Taken at face value, this result would lead to an absurd conclusion that in order to observe neutrino oscillations e.g. in reactor or accelerator neutrino experiments one would not need far detectors at all – it would be sufficient to put the detector immediately next to the neutrino source and just wait long enough.

In order to solve this problem, the "time to space conversion" procedure of Eq. (12) is usually invoked. Let us look at this procedure more carefully. One usually tries to justify it by the fact that neutrinos are ultra-relativistic, i.e. they propagate with nearly the speed of light. However, this is not the most important assumption behind Eq. (12). The same argument could be made even for non-relativistic neutrinos, provided that the different mass eigenstates composing a given flavour state move with nearly the same speed $v \ll 1$ (for which they would have to be nearly degenerate in mass). In that case one would merely have to replace Eq. (12) by $L \approx vt$, and the rest of the derivation of the oscillation probability would be essentially the same. What is much more important is that Eq. (12) (as well as its modified version $L \approx vt$) is only valid for *point-like particles moving along classical trajectories*. But the notion of a point-like particle is just the opposite of that of a plane wave! So, one tries to combine two incompatible approaches in this case.

Similarly, same energy assumption based on the evolution of neutrino flavour only in space cannot account for the fact that neutrinos are produced and detected at certain times.

So, the question is: How can two different and wrong assumptions (same E and same p) result in the same (and correct) expression for the neutrino oscillation probability? To answer this question, it is necessary to consider a wave packet approach to neutrino oscillations.

3 Wave Packet Approach: The Basics

In quantum theory localized particles are described by wave packets: instead of a plane wave one considers superpositions of plane waves with a momentum spread σ_p around a central momentum $\vec{p}_0$. This is a consequence of the Heisenberg uncertainty relation: a state localized within a spatial region σ_x is characterized by a momentum uncertainty $\sigma_p \gtrsim 1/\sigma_x$. Therefore it cannot be described by a plane wave, for which $\sigma_p = 0$. In the wave packet approach the

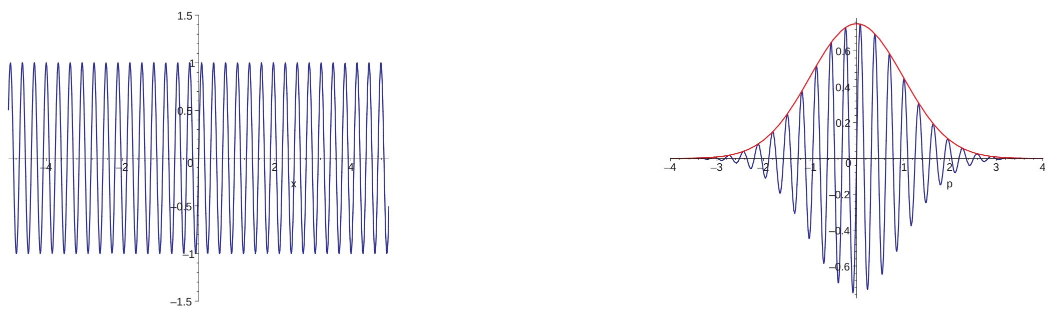


Figure 1 – Schematic representation of plane wave (left panel) and wave packet (right panel).

coordinate-space wave function of a free particle of mass m_i is

$$\Psi_i(\vec{x}, t) \;=\; \int \frac{d^3p}{(2\pi)^3}\, f_{\vec{p}_0}(\vec{p})\, e^{i\vec{p}\vec{x} - iE_i(p)t}\,, \tag{18}$$

where $f_{\vec{p}_0}(\vec{p})$ is the momentum distribution amplitude with a peak at $\vec{p} = \vec{p}_0$ and a momentum width σ_p, and $E_i = \sqrt{\vec{p}^{\,2} + m_i^2}$. A frequently used example is the Gaussian wave packet:

$$f_{\vec{p}_0}(\vec{p}) = \frac{1}{(2\pi\sigma_p^2)^{3/4}} \exp\left\{-\frac{(\vec{p} - \vec{p}_0)^2}{4\sigma_p^2}\right\}. \tag{19}$$

If one neglects the spreading of the wave packet, the coordinate-space wave function of such a state takes the form

$$\Psi_i(\vec{x},\, t) = e^{i\vec{p}_0\vec{x}-iE_i(p_0)t}\ \frac{1}{(2\pi\sigma_x^2)^{3/4}}\exp\left\{-\frac{(\vec{x}-\vec{v}_{gi}t)^2}{4\sigma_x^2}\right\}\,, \tag{20}$$

where $\vec{v}_{gi} \equiv [\partial E_i(p)/\partial\vec{p}]_{\vec{p}=\vec{p}_0} = \vec{p}_0/E_i(p_0)$ is the group velocity of the wave packet. Eq. (20) represents the plane wave corresponding to the central momentum $\vec{p}_0$ modulated by the Gaussian coordinate-space envelope factor of the width $\sigma_x = 1/(2\sigma_p)$ (see Fig. 1). The quantity σ_x is therefore the spatial length of the wave packet. Note that though the time t and the spatial coordinate $\vec{x}$ are, strictly speaking, independent, the probability of finding the particle $|\Psi_i(\vec{x},t)|^2$, which reaches its maximum at $\vec{x} = \vec{v}_{gi}t$, quickly decreases when $|\vec{x}-\vec{v}_{gi}t|$ starts exceeding σ_x. This holds true for the wave function of any localized state, not just for Gaussian wave packets.

In the wave packet approach the evolved state for a neutrino which was produced as ν_α is

$$|\nu(\vec{x},t)\rangle \;=\; \sum_i U_{\alpha i}^*|\nu_i(\vec{x},t)\rangle \;=\; \sum_i U_{\alpha i}^*\,\Psi_i(\vec{x},t)|\nu_i\rangle\,, \tag{21}$$

where the wave function $\Psi_i(\vec{x},t)$ of the ith neutrino mass eigenstate is given in Eq. (18). This is to be compared with the expression for the evolved neutrino state in the plane wave approach (7). The state of the detected neutrino ν_β, on which the evolved state has to be projected in order to find the transition amplitude, should also be described by a localized wave packet. The wave packet approach to neutrino oscillations allows one to resolve many paradoxes and confusions in the oscillation theory.

3.1 Oscillation Phase and the Wave Packet Approach

How can one calculate the oscillation phase (10), now that we know that both the same energy and same momentum approaches are actually incorrect? Let us take into account that we usually deal only with highly relativistic neutrinos.[b] In this case the energy and momentum differences of the different neutrino mass eigenstates composing a given flavour state are small compared to their respective average values ($\Delta E \ll E$, $\Delta p \ll p$). One can therefore expand ΔE as

$$\Delta E = \frac{\partial E}{\partial p}\Delta p + \frac{\partial E}{\partial m^2}\Delta m^2 = v_g\Delta p + \frac{1}{2E}\,\Delta m^2\,, \tag{22}$$

where v_g is the average group velocity of the neutrino mass eigenstates. Substituting this into Eq. (10) yields

$$\Delta\phi = -(L - v_g t)\Delta p + \frac{\Delta m^2}{2E}\,t\,. \tag{23}$$

Let us examine this expression. If one adopts the (incorrect) same momentum approach, $\Delta p = 0$, the first term on the right hand side (r.h.s.) vanishes, and the oscillation phase (11) is recovered. Note, however, that the first term on the r.h.s. of (23) vanishes also when $\Delta p \neq 0$ provided that $L = v_g t$, which corresponds to the center of the neutrino wave packet. Away from the center, $L - v_g t$ does not vanish but, as was discussed above, its value cannot significantly exceed the spatial length of the wave packet σ_x. Therefore, the first term on the r.h.s. is negligibly small and Eq. (11) obtains without the unphysical "same momentum" prescription provided that

$$\sigma_x \cdot |\Delta p| \ll 1\,. \tag{24}$$

By the Heisenberg uncertainty relation $\sigma_x \sim 1/\sigma_p$, and therefore Eq. (24) is equivalent to

$$|\Delta p| \ll \sigma_p\,. \tag{25}$$

[b] The following arguments also apply to moderately relativistic as well as to non-relativistic neutrinos provided that they are nearly degenerate in mass.

In addition, from $|L - v_g t| \lesssim \sigma_x$ it follows that one can replace $t \to L/v_g$ in the second term on the r.h.s of Eq. (23) provided that σ_x is negligibly small compared to the neutrino oscillation length. This yields the standard oscillation phase $\Delta\phi = [\Delta m^2/(2p)]L$.

Quite similarly one can show that the correct oscillation phase leading to the standard oscillation probability (6) can be obtained without the "same energy" assumption. Expressing Δp from Eq. (22) and substituting it into Eq. (10), one finds for the oscillation phase

$$\Delta\phi = -\frac{1}{v_g}(L \; - \; v_g\, t)\Delta E \; + \; \frac{\Delta m^2}{2p}\, L\,, \tag{26}$$

which is equivalent to Eq. (23). In the limit $\Delta E \to 0$ the first term on the r.h.s. of this equation vanishes and the results of the "same energy" approach are recovered. However, this term can also be neglected even when $\Delta E \neq 0$, provided that $(\sigma_x/v_g)|\Delta E| \ll 1$. As the spatial length of the wave packet satisfies $\sigma_x \sim 1/\sigma_p \simeq v_g/\sigma_E$ where σ_E is the energy uncertainty of the neutrino state,[c] we find that the first term on the r.h.s. of Eq. (26) can be neglected when

$$|\Delta E| \ll \sigma_E\,. \tag{27}$$

We can now answer the question raised at the end of Section 2.1. The wrong "same energy" and "same momentum" assumptions lead to the correct oscillation probability (6) because

- Neutrinos are relativistic with $|\Delta E| \ll E$, $|\Delta p| \ll p$, so that the expansion in Eq. (22) is justified.
- In most (if not all) situations of practical interest, energy and momentum differences ΔE and Δp are small compared to the intrinsic QM energy and momentum uncertainties of the neutrino state, i.e. conditions (25) and (27) are satisfied.

As we shall see, Eqs. (25) and (27) are essentially the coherence conditions for neutrino production and detection.

4 When Are Neutrino Oscillations Observable? The Role of QM Uncertainties

Quantum mechanics tells us that no particle can have precisely defined values of energy and momentum – these quantities always have some intrinsic uncertainties. This is due to the fact that particles are always localized in space and time. In particular, the processes in which the particles are produced and which actually determine their properties are always confined to finite space-time intervals. The QM uncertainty principles relate the momentum and energy uncertainties of a particle, σ_p and σ_E, to the spatial extension and the duration of its production process, σ_X and σ_t: [d]

$$\sigma_p \sim \sigma_X^{-1}\,, \qquad \sigma_E \sim \sigma_t^{-1}\,. \tag{28}$$

Usually, energy and momentum uncertainties of the particles are extremely small compared to their energies and momenta themselves; therefore, in most situations these uncertainties can be safely neglected. This is, however, not justified when neutrino oscillations are considered. The reason is that the neutrino energy and momentum uncertainties, as tiny as they are, are crucially important for the oscillation phenomenon – without them the oscillations just would not occur. Indeed, as discussed above, if neutrinos were produced, for example, with no momentum uncertainty, this would mean that their source was completely delocalized in space, and therefore neutrino oscillations as a function of the distance L between the neutrino source and detector would be unobservable. Similar arguments apply to the neutrino energy uncertainty.

[c] The relation $\sigma_E \simeq v_g \sigma_p$ follows from the mass-shell condition $E^2 = \vec{p}^{\,2} + m^2$.

[d] Strictly speaking, the QM uncertainty relations read $\sigma_p \geq (2\sigma_X)^{-1}$, $\sigma_E \gtrsim (\sigma_t)^{-1}$, that is, Eq. (28) should actually contain inequalities rather than the $\sim$ symbol. However, except in very special cases of little practical interest, the relations in Eq. (28) apply.

4.1 Coherence Conditions for Neutrino Production and Detection

The fact that too accurate measurement of neutrino energy and momentum would destroy neutrino oscillations was first demonstrated by Kayser [10]. Assume that by measuring the energies and momenta of all particles participating in a neutrino production process we reconstructed the energy E and momentum p of the produced neutrino with some accuracy. The errors in the determination of E and p cannot be smaller than the intrinsic QM uncertainties σ_E and σ_p related to the space-time localization of the production process. Assuming that these uncertainties are independent, from the mass-shell relation $E^2 = \vec{p}^{\,2} + m^2$ one can then infer the squared mass of the emitted neutrino with the minimum uncertainty

$$\sigma_{m^2} = [(2E\sigma_E)^2 + (2p\sigma_p)^2]^{1/2}\,. \tag{29}$$

If this minimum uncertainty is large compared to the difference between the squared masses of different neutrino mass eigenstates, i.e. $\sigma_{m^2} \gg |\Delta m^2|$, it is in principle impossible to find out which neutrino mass eigenstate was produced. This means that different neutrino mass eigenstates are produced coherently; what is actually emitted is a flavour eigenstate (5), which is a coherent linear superposition of the neutrino mass eigenstates.

Conversely, for $\sigma_{m^2} \lesssim |\Delta m^2|$ one can find out which neutrino mass eigenstate has actually been produced; this means that different mass eigenstates cannot be produced coherently. As neutrino oscillations are a result of interference of the amplitudes corresponding to different neutrino mass eigenstates, the absence of their coherence means that no oscillations will take place. The flavour transition probabilities will then correspond to averaged neutrino oscillations.

Coherence conditions for neutrino production and detection processes can also be formulated in configuration space. It was demonstrated by Kayser [10] that as soon as σ_E and σ_p become sufficiently small to allow determination of the neutrino mass at neutrino production, the uncertainty in the coordinate of the neutrino production point becomes larger than the oscillation length $l_{\rm osc} = 4\pi p/\Delta m^2$, and so the oscillations get washed out due to the averaging over this coordinate. Similar arguments apply to neutrino detection. Thus, the production and detection processes cannot discriminate between different neutrino mass eigenstates (which is a necessary condition for the observability of neutrino oscillations) only when the uncertainties in the neutrino emission and absorption coordinates are sufficiently small, i.e. when the processes of neutrino production and detection are sufficiently well localized. For this reason the conditions of coherent neutrino production and detection are sometimes called the localization conditions.

Coherence of neutrino production and detection in the configuration-space formulation was also considered in Refs. [11,12,13]. Here I discuss it following Ref. [13].

The 4-coordinate of the neutrino production point has an intrinsic uncertainty (δt, $\delta\vec{x}$) related to the finite space-time extension of the production process. This leads to the fluctuations $\delta\phi_{osc} \equiv \delta(\Delta\phi)$ of the oscillation phase (10):

$$\delta\phi_{osc} = \Delta E\cdot\delta t - \Delta\vec{p}\cdot\delta\vec{x}\,. \tag{30}$$

For neutrino oscillations to be observable, these fluctuations must be small – otherwise the oscillations will be washed out upon averaging of the oscillation phase over the 4-coordinate of neutrino production. That is, a necessary condition for the observability of neutrino oscillations is

$$|\Delta E\cdot\delta t - \Delta\vec{p}\cdot\delta\vec{x}| \ll 1\,. \tag{31}$$

Barring accidental cancellations between the two terms in (31), one can rewrite it as

$$|\Delta E\cdot\delta t| \ll 1\,, \qquad |\Delta\vec{p}\cdot\delta\vec{x}| \ll 1\,. \tag{32}$$

Now, the fluctuations of the neutrino emission time and coordinate are limited by the temporal extension of the production process and the spatial size of the production region:

$$|\delta t| \lesssim \sigma_t\,, \qquad |\delta\vec{x}| \lesssim \sigma_X\,. \tag{33}$$

Taking into account Eq. (28), from (32) we therefore obtain

$$|\Delta E| \ll \sigma_E\,, \qquad |\Delta p| \ll \sigma_p\,. \tag{34}$$

These conditions actually have a simple meaning: Different neutrino mass eigenstates can be emitted coherently and compose a flavour state only if their intrinsic QM energy and momentum uncertainties, σ_E and σ_p, are sufficiently large to accommodate their differing energies and momenta. Similar considerations apply to neutrino detection: energy and momentum uncertainties related to the space-time localization of the detection process must be large enough to preclude determination of the neutrino mass, or else the oscillations will be unobservable. If by σ_E we understand the smallest between of the energy uncertainties associated with neutrino production and detection (and similarly for the momentum uncertainty σ_p), conditions (34) will ensure coherence of both neutrino production and detection processes. Note that these conditions coincide with those in Eqs. (25) and (27) which allowed us to obtain the standard oscillation phase from the general expression (10).

While the coherent production condition in Eq. (31) is obviously Lorentz invariant,[e] the conditions in Eq. (34) are not. They follow from (31) only in the absence of cancellations between the $\Delta E \cdot \delta t$ and $\Delta\vec{p}\cdot\delta\vec{x}$ terms. It can be shown that even if this no-cancellation requirement is met in a reference frame K, it may be violated in reference frames moving with the speed $u \approx 1$ with respect to K provided that $1-u$ is small enough [13]. In such frames the inequalities in Eq. (34) do not play the role of the coherent production/detection conditions, and Eq. (31) should be used instead. In what follows I will be assuming that the no-cancellation condition is met, so that the inequalities in Eq. (34) do play the role of conditions for coherent neutrino production and detection.

4.2 Propagation Coherence and Decoherence

For neutrino oscillations to be observable it is not sufficient that the neutrino production and detection processes be coherent: In addition, coherence must not be (irreversibly) lost during neutrino propagation from its source to the detector.

How could a loss of neutrino coherence in transit from the source do the detector occur? This question was first studied by Nussinov in what appears to be the first publication on the wave packet approach to neutrino oscillations [14]. The wave packets of the different neutrino mass eigenstates composing a produced flavour eigenstate propagate with different group velocities. This is because these velocities, $\vec{v}_{gi} \equiv \partial E_i/\partial\vec{p}_i = \vec{p}_i/E_i$, depend on the neutrino mass m_i. Due to the difference of the group velocities Δv_g, over a time t the centers of the wave packets of the different neutrino mass eigenstates separate by the distance $\Delta v_g t$. Once this distance exceeds the spatial length of the wave packets σ_x, the wave packets of different neutrino mass eigenstates cease to overlap and so lose their coherence. In this case neutrino oscillations cannot be observed. This can be seen from the fact that, once the wave packets of the different mass eigenstates have separated in space, one can in principle discriminate between them at detection, e.g., by making use of the time-of-flight measurement technique.

The coherence time and coherence length can be found from the conditions

$$\Delta v_g \cdot t_{\rm coh} \;\simeq\; \sigma_x\,; \qquad\quad l_{\rm coh} \;\simeq\; v_g t_{\rm coh}\,, \tag{35}$$

where in the second equality v_g denotes the average group velocity of the mass eigenstates (recall that we are assuming $\Delta m^2 \ll E^2$, so that $\Delta v_g \ll v_g$). For ultra-relativistic neutrinos $\Delta v_g \simeq \frac{\Delta m^2}{2E^2}$, and Eq. (35) yields [14]

$$l_{\rm coh} \simeq \frac{v_g}{|\Delta v_g|}\sigma_x \simeq \frac{2E^2}{|\Delta m^2|}\,\sigma_x\,. \tag{36}$$

[e] Indeed, both the oscillation phase and its variation, being products of two 4-vectors, are Lorentz invariant.

Neutrino oscillations can only be observed at the distances L from the neutrino source satisfying $L \ll l_{\rm coh}$. Although the lengths of the neutrino wave packets σ_x are usually microscopically small, the coherence length $l_{\rm coh}$ is macroscopic and very large because of the huge factor $2E^2/\Delta m^2$ multiplying σ_x in the expression for $l_{\rm coh}$.

An interesting point, first made by Kiers, Nussinov and Weiss [15,16], is that even if the wave packets of the different neutrino mass eigenstates composing an emitted flavour state have separated on their way between the neutrino source and detector, their coherence may be restored at neutrino detection. The point is that any detection process is not instantaneous – it takes a finite time $\sigma_{t\,det}$ (which is related to the ultimate energy resolution of detection $\sigma_{E\,det}$ by $\sigma_{t\,det} \sim \sigma_{E\,det}^{-1}$). Assume that the elementary detection process lasts long enough, so that the wave packets of the different neutrino mass eigenstates, which have separated during the neutrino propagation, arrive at the detector before the detection is over. Then their detection amplitudes may add coherently and interfere, leading to observable flavour oscillations. Possible restoration of propagation coherence at detection can be automatically taken into account if in the expression for the coherence length in Eq. (36) by σ_x we understand an "effective" length of the neutrino wave packet, defined as $\sigma_x = v_g/\sigma_E$ with σ_E being the smaller between the QM energy uncertainties associated with neutrino production and detection. In particular, in the limit $\sigma_{E\,det} \to 0$ the coherence length formally becomes infinite, i.e. decoherence by wave packet separation never occurs. Note, however, that for too small $\sigma_{E\,det}$ the condition of coherent detection of different neutrino mass eigenstates (34) may be violated. So, the issue of compatibility of the different coherence conditions should be considered.

4.3 Are Different Coherence Constraints Compatible?

We have found that there are two types of coherence conditions that have to be satisfied in order for neutrino oscillations to be observable: (i) coherence of neutrino production and detection and (ii) coherence of neutrino propagation. Before proceeding to discuss their compatibility, let me make the following point. One can show that under very general assumptions the second condition in (34) follows from the first one and so is actually redundant [13]. That is, the first condition in (34) by itself ensures coherence of neutrino production and detection.

The production/detection and the propagation coherence conditions, $\Delta E \ll \sigma_E$ and $L \ll l_{\rm coh}$, both put upper limits on the neutrino mass squared difference Δm^2:

$$\Delta E \sim \frac{\Delta m^2}{2E} \ll \sigma_E\,, \qquad \frac{\Delta m^2}{2E^2} L \ll \sigma_x \simeq v_g/\sigma_E\,. \tag{37}$$

However, when expressed as constraints on σ_E, they read (taking into account that $v_g \approx 1$)

$$\frac{\Delta m^2}{2E} \ll \sigma_E \ll \frac{2E^2}{\Delta m^2}\frac{1}{L}\,, \tag{38}$$

that is, they constrain σ_E both from above and from below. A natural question then is: Are these constraints compatible with each other? With decreasing Δm^2 the left hand side (l.h.s.) of Eq. (38) decreases, while its r.h.s. increases; so it is clear that the smaller Δm^2, the easier it is to satisfy the conditions in Eq. (38).[f] For the two conditions in Eq. (38) to be compatible, its l.h.s. must be small compared to its r.h.s., which gives

$$2\pi \frac{L}{l_{\rm osc}} \ll \frac{2E^2}{\Delta m^2} \;\; (\gg 1)\,. \tag{39}$$

It should be stressed that this condition is necessary but in general not sufficient for a mixed neutrino state to be coherently produced, maintain its coherence over the distance L and then

[f] By the way, this may already give us a hint on what the answer to the question "Do charged leptons oscillate?" should be.

be coherently detected: it only ensures the consistency of the two conditions in Eq. (38), but not their separate fulfilment. Note that from the rightmost inequality in Eq. (38) it follows that the maximum number of observable oscillations $(L/l_{\rm osc})_{\rm max}$ cannot exceed

$$\frac{l_{\rm coh}}{l_{\rm osc}} \sim \frac{1}{2\pi}\frac{E}{\sigma_E}\,. \tag{40}$$

In reality, the observable number of oscillations is much smaller: it is obtained from the r.h.s. of Eq. (40) by replacing $\sigma_E \to \delta E$, where δE is the energy resolution of the detector which is usually much larger than the ultimate QM energy uncertainty σ_E. This puts an upper limit on the baselines at which the oscillations can be observed. For longer baselines, the flavor transition (and survival) probabilities will correspond to Eq. (6) with all the oscillatory terms replaced by their averaged values.

4.4 Are They Actually Satisfied?

Are the coherence conditions normally satisfied in neutrino oscillation experiments? For oscillations between the usual flavour-eigenstate neutrinos ν_e, ν_μ and ν_τ in the 3-flavour scheme, the coherent production and detection conditions are satisfied with a large margin in all cases of practical interest. This is a consequence of the tininess of the masses of their mass-eigenstate counterparts ν_1, ν_2 and ν_3 and is now firmly established by the positive results of the experiments on atmospheric, reactor and accelerator neutrino oscillations.

A rather obvious though not widely recognized is the fact that even non-observation of neutrino flavour transitions in experiments with the detector situated too close to the neutrino source (as it was the case e.g. in 'prehistoric' reactor and accelerator neutrino experiments) is a direct consequence of and a firm evidence for coherence of neutrino production and detection. Indeed, if coherence was violated, i.e., if different neutrino mass eigenstates were emitted or absorbed incoherently, one would have observed a suppression of the original neutrino flux (in the disappearance experiments) or an emergence of "wrong-flavour" neutrinos (in the appearance experiments) corresponding to the averaged oscillation probabilities.[g] As the leptonic mixing angles are relatively large (especially θ_{23} and θ_{12}), effects of such flavour transitions would have been quite sizeable.

How about the propagation coherence condition? For 3-flavour oscillations between ν_e, ν_μ and ν_τ it is satisfied with a large margin for all terrestrial experiments. In particular, from the atmospheric neutrino experiments we know that the coherence holds over macroscopic distances as large as about 10,000 km. At the same time, for astrophysical and cosmological neutrinos which propagate enormous distances before reaching the earth, coherence is lost. Thus, solar and supernova neutrinos arrive at terrestrial detectors as incoherent mixtures of the mass eigenstates rather than as flavour eigenstate neutrinos. Cosmic background neutrinos should also at present be in mass rather than in flavour states.

The situation with coherence of neutrino flavour transitions may, however, be different if relatively heavy predominantly sterile neutrinos exist. Note that heavy neutrinos with masses in the eV – keV – MeV (and even GeV) ranges are now being actively discussed in connection with some hints from short-baseline accelerator experiments (LSND, MiniBooNE), reactor neutrino anomaly, anomaly in gallium radioactive source experiments, keV sterile neutrinos as dark matter, pulsar kicks, leptogenesis via neutrino oscillations, supernova r-process nucleosynthesis, unconventional contributions to $2\beta 0\nu$ decay, etc. [17]. For the corresponding large mass squared differences the production/detection and propagation coherence conditions may be violated, leading to important implications. Therefore in situations when large Δm^2 may play a role the fulfilment of the coherence conditions should be carefully examined in each particular case.

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[g] For example, in the 2-flavour case the survival probability at short baselines $L \ll l_{\rm osc}$ is $P_{\alpha\alpha} = (\sin^2\theta + \cos^2\theta)^2 = 1$ in the coherent case and $P_{\alpha\alpha} = \sin^4\theta + \cos^4\theta < 1$ if coherence is strongly violated at neutrino production or detection.

5 Wave Packet Approach: Gaussian Wave Packets

So far I have been discussing neutrino production/detection and propagation coherence as necessary conditions for observability of neutrino oscillations in a rather qualitative way, basing on some very general arguments. Can we see how this works in an explicit calculation? As I discussed earlier, for this one needs to resort to wave packets.

To the best of my knowledge, the first complete calculation of the oscillation probability in the wave packet framework was performed by Giunti, Kim and Lee [18,19], though simplified studies can also be found in some earlier papers (see Section 12 below). Describing neutrinos by Gaussian wave packets, Giunti et al. found the following expression for the probability of $\nu_\alpha \to \nu_\beta$ oscillations:

$$P_{\alpha\beta}(L,E) = \sum_{i,k} U_{\alpha i} U_{\beta i}^* U_{\alpha k}^* U_{\beta k} e^{-i(\Delta m_{ik}^2/2E)L}\, e^{-[\Delta E_{ik}^2/8\sigma_E^2]-[L/(l_{\rm coh})_{ik}]^2}\,. \tag{41}$$

Here the indices i,k correspond to neutrino mass eigenstates and

$$(l_{\rm coh})_{ik} = 2\sqrt{2}\frac{2E^2}{|\Delta m_{ik}^2|}\sigma_x \tag{42}$$

is the coherence length (which is to be compared with Eq. (36) found from qualitative arguments). The quantity σ_E is the effective neutrino energy uncertainty which is expressed through the energy uncertainties related to neutrino production and detection as

$$\frac{1}{\sigma_E^2} = \frac{1}{\sigma_{E\,prod}^2} + \frac{1}{\sigma_{E\,det}^2}\,, \tag{43}$$

i.e. it is dominated by the smaller of the two. The last exponential factor on the r.h.s. of Eq. (41),

$$e^{-[\Delta E_{ik}^2/8\sigma_E^2]-[L/(l_{\rm coh})_{ik}]^2}\,, \tag{44}$$

takes into account possible decoherence effects. It strongly suppresses the off-diagonal terms in the summand of Eq. (41) when either $|\Delta E_{ik}| \gg \sigma_E$, which means violation of production or detection coherence, or $L \gg (l_{\rm coh})_{ik}$, which implies (irreversible) decoherence by wave packet separation. The suppression of the off-diagonal terms in (41) would mean that all the oscillatory terms in the oscillation probability are effectively replaced by their averages, giving

$$P_{\alpha\beta}(L,E) \to \bar{P}_{\alpha\beta} \equiv \sum_i |U_{\alpha i}|^2 |U_{\beta i}|^2\,, \tag{45}$$

which is L- and E-independent. Note that the same result is obtained from the standard expression for the oscillation probability in Eq. (6) upon averaging out all its oscillatory terms.

On the other hand, if the production/detection and propagation coherence conditions are satisfied for all (i,k), one can replace the last exponential factor in Eq. (41) by unity; the resulting probability of flavour transitions then coincides with the standard probability of neutrino oscillations in vacuum (6).[h]

6 Wave Packet Approach and the Normalization Problem

In deriving the oscillation probabilities in the wave packet approach, one encounters the following problem: by using the standard normalization of the wave packets in coordinate space $\int d^3x |\Psi_i(t,\vec{x})|^2 = 1$ (or equivalently, in momentum space, $\int [d^3p/(2\pi)^3] |f_{\vec{p}_0}(\vec{p})|^2 = 1$) one does not get the correct normalization for the oscillation probability. Instead, the result differs from

[h] Here and in most of the following text I neglect the difference between p and E which is justified for ultra-relativistic neutrinos. The exception is Section 10 where the Lorentz invariance issues are discussed.

the standard oscillation probability by a constant factor. The usual way to circumvent this difficulty is to introduce arbitrary normalization factors for the wave functions of the produced and detected neutrino states and then fix them at the end of the calculation by imposing on the oscillation probabilities the unitarity condition

$$\sum_{\beta} P_{\alpha\beta}(L,E) = 1\,. \tag{46}$$

Although it works, this is an *ad hoc* procedure which looks rather unsatisfactory. Can one avoid it by using a different normalization of the neutrino wave functions?

It turns out that the answer to this question is negative: no separate (i.e. independent) normalization of the wave functions of the produced and detected neutrino states would result in the correctly normalized $P_{\alpha\beta}(L,E)$. For example, if $f^{P}_{\vec{p}_0}(\vec{p})$ and $f^{D}_{\vec{p}_0}(\vec{p})$ are the momentum-space wave functions of the produced and detected neutrinos (with the indices P and D standing for production and detection), the correct normalization of the oscillation probability is only obtained provided the condition

$$\int \frac{d^3p}{(2\pi)^3} |f^{P}_{\vec{p}_0}(\vec{p})|^2 |f^{D}_{\vec{p}_0}(\vec{p})|^2 = 1 \tag{47}$$

is satisfied. That is, in order to obtain the correctly normalized oscillation probability, one needs to impose a correlated normalization condition for the wave functions of the produced and detected neutrino states; no separate normalization of them would do the job.

It is, actually, not difficult to understand why this happens. The problem is related to the wave packet description of the process itself and not to neutrino oscillations: we would have encountered a similar problem when considering neutrino production and subsequent detection in the wave packet picture even if there existed just one neutrino species in Nature, i.e. if the oscillations were absent. The point is that in the wave packet approach each individual particle (and not just an ensemble of them) is characterized by a spectrum of momenta (and energies). For the produced and detected neutrinos, the corresponding spectra are dictated by the character and the properties of the production and detection processes, such as their localization. The production and detection processes are different, and so are the corresponding momentum spectra. The detection efficiency therefore depends on the degree of overlap of these spectra, which is given by the integral on the l.h.s. of Eq. (47). This just reflects energy and momentum conservation. In particular, in the special case when the overlap is absent, i.e. when the momentum modes corresponding to detection are absent from the spectrum of the produced wave packets (which happens when the energy threshold of the detection process is above the maximum energy of the produced particles), there will be no detection at all.

Coming back to neutrino oscillations, kinematic prohibition of neutrino detection would not, of course, mean that neutrinos do not oscillate. On the other hand, even if the overlap of the spectra of the produced and detected neutrinos is perfect, i.e. the integral on the l.h.s. of Eq. (47) takes its maximum possible value, this does not solve the normalization problem [20]. The point is that the overlap integral is simply not a part of the oscillation probability. Obviously, the oscillation probability itself should be independent of the efficiency of neutrino detection. In calculating $P_{\alpha\beta}(L,E)$, all the details of the neutrino production and detection processes should be factored out and removed. Normalization of $P_{\alpha\beta}(L,E)$ by imposing the unitarity constraint does just this. I will discuss this point in more detail in the next Section, where the related issue of universality of the oscillation probabilities is considered.

7 Universal Oscillation Probabilities?

The standard oscillation probability (6) depends only on neutrino energy E and the distance from the source L, but not on the processes in which the neutrinos were produced and detected.

Under what conditions is this justified? That is, when are neutrino oscillations actually described by a universal (i.e. production– and detection–independent) probability?

Strictly speaking, in general one should consider neutrino production, propagation and detection as a single process, as only the probability of the complete process, $\Gamma_{\alpha\beta}(L,E)$, is directly related to measurable quantities in oscillation experiments. Under certain conditions $\Gamma_{\alpha\beta}(L,E)$ can be factorized as

$$\Gamma_{\alpha\beta}(L,E) = j_\alpha(E)\, P^{\rm prop}_{\alpha\beta}(L,E)\sigma_\beta(E)\,, \tag{48}$$

where $j_\alpha(E)$ is the flux of the produced ν_α, $P^{\rm prop}_{\alpha\beta}(L,E)$ is the probability of neutrino propagation between its source and detector, which takes into account possible $\nu_\alpha \to \nu_\beta$ transitions and also includes a geometrical factor describing neutrino flux suppression with increasing distance L from the source, and $\sigma_\beta(E)$ is the detection cross section for ν_β. If such a factorization is possible, one can find the oscillation probability $P_{\alpha\beta}(L,E)$ by dividing $\Gamma_{\alpha\beta}(L,E)$ by $j_\alpha(E)$, $\sigma_\beta(E)$ and by the trivial geometric factor of neutrino flux suppression with distance.[i] If, however, the factorization (48) turns out to be impossible, the production– and detection–independent oscillation probability cannot even be defined. In this case one should instead consider the probability $\Gamma_{\alpha\beta}(L,E)$ of the overall neutrino production–propagation–detection process.

So, when is the factorization (48) actually possible? It can be shown [12,20] that for such a factorization to take place the following conditions must be satisfied:

(a) Neutrinos are ultra-relativistic or quasi-degenerate in mass;

(b) Neutrino production, propagation and detection processes are coherent, i.e. they do not allow one to discriminate between different neutrino mass eigenstates.

These conditions can be easily understood. The factorization (48) only takes place when all the three processes – neutrino production, propagation and detection – are independent of each other. If condition (a) is violated, the composition of the produced neutrino state ν_α in terms of the mass eigenstates ν_i will not be given by Eq. (5), but will rather depend sensitively on the neutrino masses m_i in a way that depends on the kinematics of the production process. As the flavour transition probabilities depend on the composition of the produced neutrino state, neutrino production and propagation processes will not be independent in this case. If condition (b) is not met, the probabilities of flavour transformations will depend on the degree of coherence violation at neutrino production and/or detection, so that there will again be no independence of neutrino production, propagation and detection.

If we compare conditions (a) and (b) for the universal oscillation probability to exist with the discussed earlier conditions for neutrino oscillations in vacuum to be described by the standard oscillation probability, we will find that they exactly coincide. This is an important point: *whenever the universal (production– and detection–independent) oscillation probability makes sense at all, it is given by the standard oscillation formula (6).*

What happens if the coherence conditions are partially or completely violated? In this case one can in principle introduce an effective oscillation probability by *defining* it as the probability of the overall process $\Gamma_{\alpha\beta}(L,E)$ divided by the flux of the produced neutrinos $j_\alpha(E)$, detection cross section $\sigma_\beta(E)$ and the geometric factor of neutrino flux suppression with distance. Such an effective oscillation probability will obviously be non-universal. It can be shown that, if condition (a) above is satisfied and in addition either neutrino production or detection is coherent, the so defined effective oscillation probability will be correctly normalized. An example of such a production– and detection–dependent effective oscillation probability is given by Eq. (41).

More detailed discussion of the conditions for the existence of the universal and/or correctly normalized oscillation probability can be found in Ref. [20].

[i] Note that this procedure will automatically lead to the correctly normalized oscillation probability because it rids the transition probability of the details of the neutrino production and detection processes.

8 Small Corrections?

We know that the standard formula for the oscillation probability (6) is correct and works well in most cases of practical interest, provided that matter effects on neutrino oscillations are absent or can be neglected. However, it is obviously not exact, and so one may be tempted to look for (presumably) small corrections to it. For example, one might look for the corrections to the standard oscillation phase due to the next terms in the relativistic expansion of neutrino energy $(p_i^2 + m_i^2)^{1/2} \approx p_i + m_i^2/(2p_i) + \ldots$. The next-to-leading order contribution to the oscillation phase $\phi_{\rm osc}$ will then be proportional to Δm^4. At first sight, this may make sense. However, the correction will only become noticeable at baselines L at which its contribution to the oscillation phase becomes comparable to 1. It is easy to see that for such distances the leading order term in the phase is of order $E^2/m_\nu^2 \ggg 1$; this means that the oscillations are already in the complete averaging regime, and any correction of order one to the oscillation phase are irrelevant.

9 Are Neutrino Oscillations Compatible With Energy-Momentum Conservation?

In quantum theory, the rates of processes are calculated by making use of the generalized Fermi's golden rule

$$\Gamma \;=\; \prod_i \frac{1}{(2E_i)} \int \prod_f \frac{d^3 p_f}{(2\pi)^3\, 2E_f} |M_{fi}|^2 (2\pi)^4 \delta^4\Big(\sum_f p_f - \sum_i p_i\Big)\,, \tag{49}$$

where the factor $\delta^4(\sum_f p_f - \sum_i p_i)$ ensures energy-momentum conservation. In particular, it is used to calculate neutrino production rates and detection cross sections. However, if one applies it e.g. to neutrino production, one might conclude that the neutrino 4-momentum $p = (E, \vec{p})$ can be determined from the 4-momenta of all the other particles participating in the production process. But then from the on-shell relation $E^2 = \vec{p}^{\,2} + m^2$ one would be able to find the neutrino mass, which would mean that the produced neutrino is a mass eigenstate and not a flavour one. This would imply that neutrinos cannot oscillate. Similar arguments could be made for neutrino detection.

Thus, a dichotomy arises: On the one hand, energy-momentum conservation is, to the best of our knowledge, an exact law of nature. On the other hand, applying energy and momentum conservation to neutrino production or detection would apparently make neutrino oscillations impossible, in contradiction with experiment.

This caused a significant confusion in the literature. As I discussed above, the resolution of the paradox comes from the observation that particles participating in neutrino production and detection processes are localized in space and time, and therefore their energies and momenta have intrinsic QM uncertainties. Although these uncertainties are usually very small, they cannot be ignored when neutrino oscillation are considered. In other words, one has to take into account that the states of these particles are not exact eigenstates of energy and momentum. This does not mean, of course, that energy and momentum conservation laws are violated.

10 Lorentz Invariance of the Oscillation Probabilities

The probabilities of neutrino flavour transitions must not depend on the Lorentz frame in which the oscillations are considered, i.e. must be Lorentz invariant. Can we see that this is indeed the case? In particular, is the standard oscillation probability (6) invariant with respect to Lorentz transformations?

In addition to its dependence on the neutrino mass squared differences and the parameters of the leptonic mixing matrix U which are universal constants, the expression in eq. (6) depends only on the distance L from the neutrino source and the mean momentum of the neutrino state p through their ratio L/p. Is this quantity Lorentz invariant? It is not difficult to show that it is, provided that the condition $L = v_g t$ is satisfied [21]. Note that the relation $L = v_g t$ is itself Lorentz covariant [12]; it essentially means that neutrinos are considered as point-like particles. I criticized the use of this approximation within the plane-wave approach, but it is perfectly legitimate to employ it in the wave packet framework provided that the length σ_x of the wave packet is small compared to the other characteristic lengths inherent to the problem. In the case of neutrino oscillations it is justified when σ_x is vanishingly small compared to the neutrino oscillation length. It can be shown that this requirement is indeed met when the coherent neutrino production and detection conditions are fulfilled [13]. As L/p is Lorentz invariant in this limit, so is the standard oscillation probability (6).[j] A more detailed discussion of the Lorentz invariance of the neutrino oscillation probability in the wave packet approach, including the case when the coherence conditions are not satisfied, can be found in [12].

11 Do Charged Leptons Oscillate?

The Lagrangian of the charged-current leptonic weak interactions is completely symmetric with respect to charged leptons and neutrinos, so why do we say that the charged leptons are produced and absorbed in these interactions as mass eigenstates (e, μ, τ), whereas neutrinos as flavour eigenstates (5) which are linear superpositions of the mass eigenstates ν_1, ν_2 and ν_3? Why not the other way around? Or why aren't both charged leptons and neutrinos produced and detected as linear superpositions of their respective mass eigenstates? After all, the mixing matrix U comes from the diagonalization of both neutrino and charged lepton mass matrices, so it is the *leptonic* mixing matrix (and not "the neutrino mixing matrix", as it is sometimes incorrectly called). A related question is: do charged leptons oscillate?

We know that neutrinos are emitted and detected as coherent linear superpositions of different neutrino mass eigenstates only when the coherence conditions for their production, propagation and detection are satisfied, and that all these conditions put upper limits on the neutrino mass squared differences Δm^2. Similar considerations apply also to charged leptons; the difference is that their mass squared differences are many orders of magnitude larger than those of neutrinos. As a result, for the charged leptons the coherence conditions are not satisfied – these particles are always produced and detected as mass eigenstates and not as coherent superpositions of the mass eigenstates. This, in particular, means that they do not oscillate [22].

This also tells us that neutrinos produced, e.g., in $\pi \to \mu\nu$ and $\pi \to e\nu$ decays oscillate even when the corresponding charged leptons are not detected. For neutrino oscillations to take place, the initially produced neutrino state has to be a flavour eigenstate – a well-defined coherent superposition of the neutrino mass eigenstates. The absence of coherence of different charged leptons ensures that in each charged pion decay event either μ or e is produced but not their linear superposition. This provides a measurement of the flavour of the associated neutrino, which is necessary for neutrino oscillations to occur [22].

12 A Bit More History

The literature on QM aspects of neutrino oscillations and, in particular, on their wave-packet description, is vast. I have already mentioned several papers before; I will now discuss them in a bit more detail and will also give a very brief overview of a few more. More detailed discussions and further references can be found in [23,24,25].

As mentioned above, neutrino oscillations were first considered in the wave packet approach by Nussinov [14]. He discussed the effects of decoherence by wave packet separation and pointed out the existence of the coherence length $l_{\rm coh} \approx \sigma_x(v_g/|\Delta v_g|)$. He also estimated the lengths of

[j] In the literature it is often stated that the neutrino oscillation probabilities in vacuum depend on L/E. However, careful derivation in the wave packet framework actually yields the dependence on L/p. While for relativistic neutrinos L/E is essentially the same as L/p, when considering the Lorentz invariance issues it is important to remember that L/p is Lorentz invariant (provided that $L = v_g t$ holds), whereas L/E is not.

the neutrino wave packets σ_x and the coherence lengths $l_{\rm coh}$ for neutrinos produced in decays of isolated nuclei ($\sigma_x \approx c/\Gamma$, where c is the speed of light and Γ is the decay width of the parent particle), as well as for neutrinos from nuclear beta decay in the interior of the sun. For the lengths of the wave packets of solar neutrinos he found $\sigma_x \approx c\tau$, where τ is the time of uninterrupted neutrino emission by the nucleus, which essentially coincides with the time between collisions significantly changing the phase of the emitter. Assuming $\rho \sim 100$ g/cm^3 and $T \sim$ 1 keV as the typical density and temperature in the solar core, for solar ^{7}Be neutrinos he found $\tau \sim 3 \times 10^{-17}$ s and $l_{\rm coh} \sim 20$ km. In his estimate of the coherence length he used $\Delta m^2 \sim 1$ eV^2; with the currently known value of this quantity he would have obtained $l_{\rm coh} \sim 3 \times 10^5$ km.

As discussed in Section 4, Kayser [10] considered the coherence conditions for neutrino production and detection and related them to the space-time localization of these processes. He also presented a simplified analytic description of neutrino oscillations in the wave packet picture.

Kobzarev, Martemyanov, Okun and Shchepkin [26] were the first to include neutrino production and detection processes in the analysis of neutrino oscillations. They used a simplified model in which the neutrino source and detected were assumed to be infinitely heavy.

The first complete derivation of the neutrino oscillation probability within the wave packet approach was given by Giunti, Kim and Lee [18,19]. They described neutrinos by Gaussian wave packets and explicitly demonstrated how the oscillations get suppressed when coherence conditions are violated.

In Ref. [27] Giunti, Kim, Lee and Lee included neutrino production and detection processes with the source and target particles localized and described by Gaussian wave packets. A similar approach was independently developed by Rich [28].

Kiers, Nussinov and Weiss [15,16] pointed out that neutrino coherence lost on the way between the source and detector due to wave packet separation can in principle be recovered at detection. In Ref. [16] they also considered neutrino production, propagation and detection as a single process in a simple model with localized neutrino source and detector.

Farzan and Smirnov [11] considered neutrino propagation decoherence in momentum space as the effect of accumulation with distance of fluctuations of the oscillation phase due to the momentum spread within the wave packet. They also demonstrated Lorentz invariance of the product $\sigma_x E$ and, on a different note, pointed out that the spatial spreading of the neutrino wave packets does not affect neutrino oscillations.

In Ref. [12] a shape-independent wave packet approach to neutrino oscillation was developed, the same energy/same momentum confusion was cleared up (see also Dolgov [29] for an earlier discussion), and Lorentz invariance of the oscillation probability in both coherent and decoherent cases was demonstrated.

In Ref. [20] the QM wave packet approach to neutrino oscillations was compared with the one based on quantum field theoretic techniques, the issue of the normalization of the oscillation probability was clarified and the conditions for the existence of a universal (production– and detection–independent) oscillation probability were found.

In Ref. [13] the question of whether non-relativistic neutrinos can oscillate and the related Lorentz invariance issues were addressed.

13 Summary and Discussion

Being a quantum-mechanical interference phenomenon, neutrino oscillations owe their very existence to the QM uncertainty relations. It is the energy and momentum uncertainties of neutrinos related to the space-time localization of their production and detection processes that allow the neutrinos to be emitted and absorbed as coherent superpositions of the states of well defined and different mass. Energy and momentum uncertainties also determine the lengths of the neutrino wave packets and are therefore crucial to the issue of the loss of coherence due to the wave packet separation. Since coherence is essential for neutrino oscillations, and particles states with intrinsic energy and momentum uncertainties are described by wave packets, a consistent derivation of the oscillations probability requires using the wave packet formalism.

That being said, it does not mean that each time we want to compute the oscillation probability for a neutrino experiment we have to resort to a full-scale wave packet calculation. The tininess of the neutrino mass means that we normally deal only with ultra-relativistic neutrinos and that in most situations the coherence conditions are satisfied with a large margin. Under these conditions the probability of flavour transitions in vacuum reduces to the well known standard oscillation probability (6), which can be safely used most of the time provided that matter effects on neutrino oscillations are either absent or can be neglected. Coherence conditions, however, may be violated if relatively heavy predominantly sterile neutrinos exist – in that case their fulfilment has to be checked on the case-by-case basis.

The standard formula for the probability of neutrino oscillations in vacuum (6) is stubbornly robust – it is not easy to find a situation in which it does not work or needs significant corrections. In addition to being perfectly accurate for ultra-relativistic neutrinos in the cases when the production/detection and propagation coherence conditions are satisfied, it can also be utilized when coherence is strongly violated – one simply has to replace all the oscillatory terms in Eq. (6) by their averages. Significant deviations from Eq. (6) can only be expected when violation of coherence is moderate; if such situations exist at all, they should be quite rare.

The idea of neutrino oscillations was put forward by Pontecorvo over 60 years ago, and more than 20 years have already passed since their discovery. The theory of neutrino oscillations has been actively advanced since the 1960s and is quite mature and developed now. Consistent application of quantum theory allowed us to resolve numerous subtle issues and apparent paradoxes of the oscillation theory. In spite of this, attempts at revising some of its basic ingredients do not cease even now, usually for no good reason. Also, oversimplified approaches to the derivation of the oscillation probability can still be often found in modern reviews, lecture notes and textbooks. While the use of such simplifications in the pioneering papers is quite understandable[k], using them in contemporary literature can hardly be justified.

Though quite mature, the theory of neutrino oscillations is in my opinion far from being closed. Over the years, many times it had appeared to us to be complete and finished, but each time this turned out to be wrong. I believe that we are still in the same situation now.

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[k] Though these simplified treatments lacked justification, they actually led to the correct results, which demonstrates a good physical intuition on the part of their authors.

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